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Transformation of multipolar source fields under a change of reference frame

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Summary. Simple and convenient formulae are derived which describe the transformation of a multipolar expansion under an arbitrary proper rotation of the reference frame. When combined with the corresponding formulae for a translation, these results show how multipolar representations of source fields transform under any proper displacement of the reference frame. Particular emphasis is placed on the seismic source problem; however, these results find applications in many other physical problems.

Introduction

The radiation field from a seismic source obtained by potential methods can be described as a superposition of multipoles (Archambeau 1968; Randall 1966). Such a representation is very general, and can be obtained for a variety of source models, including relaxation sources (e.g. Archambeau 1964, 1968; Minster 1973; Burridge 1975), dislocation sources (Burridge 1969; Brune 1970; Savage 1966), as well as for underground nuclear explosions (Archambeau 1964; Archambeau & Sammis 1970; Haskell 1967). However, in most cases a multipolar representation is easily reached only in a particular coordinate system satisfying the conditions of symmetry of the source. For example, the coordinate system used by Archambeau (1964) to represent a relaxation source has an orientation directly related to the prestress. For a propagating source (Ben Menahem 1961; Archambeau 1968), the natural spherical coordinate system to be used has a polar axis oriented along the direction of rupture propagation.

Unfortunately many wave propagation problems will be most easily solved in a spherical coordinate system with polar axis along the local vertical or in a cylindrical coordinate system when use is made of a flat earth approximation.

These circumstances raise the following problem: knowing the multipolar representation of the radiation fields in a particular coordinate system, what is the equivalent representation in a different coordinate system, obtained by rotation of the first one? A solution to this problem was obtained by Satô (1950). Unfortunately Satô's solution is somewhat bulky and certainly cumbersome for use in numerical applications. In addition, the published solution suffers from numerous misprints and possibly some confusion in the definition of the Legendre associated functions (Hobson 1931; Ferrers 1877). Indeed, the product of two rotations inverse to each other will generally not yield the original multipolar expansion if Satô's results are used in their published form. In this paper we shall derive a very simple solution as a direct application of group theory. We shall use representations of the rotation group described by Gel'fand, Minlos & Shapiro (1963) and used by Burridge (1969) and Phinney & Burridge (1973). The notation used for ultraspherical functions is that of Gel'fand *et al.*, and Jacobi polynomials will be introduced in the notation of Erdelyi (1953). The final result is obtained in a very compact polynomial form especially suitable for numerical computations. The need to be able to express a multipolar expansion in a new coordinate system obtained from the original one by pure translation arises when one considers unilaterally propagating ruptures. The most convenient system to study such sources is again chosen according to symmetry arguments and generally moves along with the rupture. Then a fixed reference frame is needed to solve wave propagation problems.

The theorem needed for this purpose is an addition theorem for spherical waves. Satô (1950) proved such a theorem in the case of a translation along the polar axis. A more general case was treated by Friedman & Russek (1954) and the results re-expressed in elegant operational form by Ben Menahem (1962). Unfortunately, as pointed out by Stein (1961), these results do not yield the correct answer in all cases. The correct forms of the addition theorems for vector spherical wave functions have been derived by Stein (1961) and Cruzan (1962) and used by Thompson (1973). Miller (1964) rederived the results for standing waves as a consequence of group theory. In an independent study, Minster (1973) derived the translation theorem for multipolar expansions, and obtained a form similar to the previously known results. We shall give in this paper a short account of the derivation and the most useful form of the results.

1 Multipolar expansions

Following Archambeau (1968), we use a harmonic potential representation of the dynamic fields generated by a seismic source. Let $\mathbf{u}(\mathbf{r}, \omega)$ be the displacement field, as a function of position $\mathbf{r}$ and frequency ω. Let $\chi_4 = \nabla \cdot \mathbf{u}$ be the dilatation and $\chi_k \mathbf{e}_k = \frac{1}{2}(\nabla \times \mathbf{u})$ be the rotation fields associated with $\mathbf{u}$. We shall denote by $\chi_\alpha(\mathbf{r}, \omega)$, $\alpha = 1, 2, 3, 4$, either the dilatation (χ_4) or any one of the three components of the rotation χ_k, $k = 1, 2, 3$. Let $k_\alpha = \omega / v_\alpha$ be the wave number associated with χ_α, where v_α is the compressional velocity if $\alpha = 4$, and the shear velocity if $\alpha = 1, 2, 3$. Since it is known that all the χ_α satisfy wave equations (e.g. Archambeau 1968), then the multipolar expansion for anyone of the potentials has the form

$$\chi(\mathbf{r}_s, \omega) = \sum_{n=0}^{\infty} h_n^{(2)}(kr_s) \sum_{m=0}^{n} (\mathrm{A}_{nm}(\omega) \cos(m\Phi_s) + \mathrm{B}_{nm}(\omega) \sin m\Phi_s) P_n^m(\cos\theta_s). \tag{1}$$

$\mathrm{A}_{nm}(\omega)$ and $\mathrm{B}_{nm}(\omega)$ will be called the multipole coefficients, and r_s, ϕ_s, θ_s are the usual spherical coordinates in a particular system, say the natural source coordinate system. Here the index α on the potentials, etc. has been suppressed for brevity. A more convenient form for equation (1) is given by

$$\chi(\mathbf{r}_s, \omega) = \sum_{n=0}^{\infty} h_n^{(2)}(kr_s) \sum_{m=-n}^{n} A_m^n(\omega)\, Y_n^m(\theta_s, \phi_s) \tag{2}$$

where $Y_n^m(\theta_s, \phi_s)$ is the normalized spherical function given by

$$Y_n^m(\theta_s, \phi_s) = (-1)^m \sqrt{\frac{(n-m)!}{(n+m)!}} \sqrt{\frac{2n+1}{2\pi}} \frac{1}{\sqrt{2}} \exp(im\phi_s) P_n^m(\cos\theta_s)$$

$$= \frac{(-1)^m}{\sqrt{2\pi}} \exp(im\phi_s) \bar{P}_n^m(\cos\theta_s). \tag{3}$$

Here $\bar{P}_n^m(\mu)$ is the normalized associated Legendre function (e.g. Jahnke & Emde* 1945), satisfying $\bar{P}_n^{-m}(\mu) = (-1)^m \bar{P}_n^m(\mu)$. Then the coefficients $A_m^n(\omega)$ are obtained by identification of (1) and (2); we get

$$A_m^n(\omega) = \begin{cases} (-1)^m \sqrt{\dfrac{(n+m)!}{(n-m)!}} \sqrt{\dfrac{\pi}{2n+1}} (\mathrm{A}_{nm} - i\mathrm{B}_{nm}) & \text{for } m > 0. \\ 2\sqrt{\dfrac{\pi}{2n+1}}\, \mathrm{A}_{no} \quad \text{for } m = 0. \\ \sqrt{\dfrac{(n+|m|)!}{(n-|m|)!}} \sqrt{\dfrac{\pi}{2n+1}} (\mathrm{A}_{n|m|} + i\mathrm{B}_{n|m|}) & \text{for } m < 0. \end{cases} \tag{4}$$

2 Rotation

Let us now denote by **R** both the rotation transforming the system S into the system G, and the matrix representing this rotation in S. That is, the components in G of a vector **v** known in S are given by

$$v_i^{(G)} = R_{ik} v_k^{(S)}.$$

In particular, if $\mathbf{e}_i^{(S)}$, $\mathbf{e}_i^{(G)}$ are the basis vectors of S and G, then R_{ik} is the i-th component of $\mathbf{e}_k^{(S)}$ in the G system. By definition the functions $Y_n^m(\theta, \phi)$ form the canonical basis in the space of spherical functions of the n-th degree. In this space the rotation **R** is represented by an operator $\mathbf{T}_R$. In the canonical basis this operator is represented by an $n \times n$ matrix; we denote the (m, k) element of this matrix by T_n^{mk}, adopting the notation in use in the geophysical literature (Phinney & Burridge 1973).

The inner product of two spherical functions, f and h, of degree n, is defined by

$$\langle f, h \rangle = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi)\, \bar{h}(\theta, \phi) \sin\theta \, d\theta \, d\phi, \tag{5}$$

where $\bar{h}(\theta, \phi)$ is the complex conjugate of $h(\theta, \phi)$. With respect to the inner product the transformation $\mathbf{T}_R$ is then unitary, that is

$$\langle \mathbf{T}_R f, \mathbf{T}_R h \rangle = \langle f, h \rangle.$$

The analytical form of T_n^{mk} is derived by Gel'fand *et al.* (1963); changing their notation

* The definition of $P_n^{-m}(\mu)$ given by Jahnke & Emde (1945, p. 114) can hardly be correct since it is in conflict, for $m = 0$, with the recursion relation given just below it.

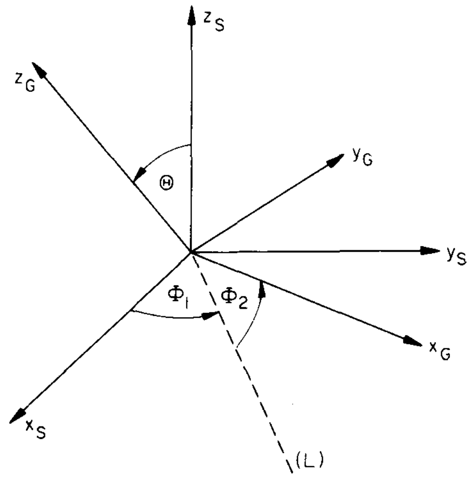

Figure 1. Definition of Euler angles. The rotations of angles Φ_1, Θ, Φ_2, are performed successively. (L) is the line of nodes, axis of the rotation Θ.

slightly, we have, in terms of the Euler angles Φ_1, Θ, Φ_2 (Fig. 1),

$$T_n^{mk} = \exp(-im\Phi_2)\, P_n^{mk}(\cos\Theta)\exp(-ik\Phi_1), \tag{6}$$

where the functions $P_n^{mk}(\mu)$ are called generalized spherical functions by Gel'fand, and are related very closely to ultraspherical functions (Erdelyi 1953); they are computed in Appendix B and are given below. Burridge (1969) and Phinney & Burridge (1973) make use of such functions to define generalized spherical harmonics. Since T_R is unitary, then (Gel'fand 1963).

$$\sum_{m=-n}^{n} |P_n^{mk}(\cos\Theta)|^2 = 1$$

Since (6) is the expression of T_n^{mk} in the canonical basis $Y_n^m(\theta, \phi)$, we can now operate with T_R on the expansion (2) to get

$$\chi(\mathbf{r}_G, \omega) = \sum_{n=0}^{\infty} h_n^{(2)}(kr_G) \sum_{m=-n}^{n} G_m^n(\omega)\, Y_n^m(\theta_G, \phi_G), \tag{7}$$

where

$$G_m^n(\omega) = \sum_{k=-n}^{n} \overline{T_n^{km}(\Phi_1, \Theta, \Phi_2)}\, A_k^n(\omega). \tag{8}$$

To complete the transformation we rewrite (8) as

$$\chi(\mathbf{r}_G, \omega) = \sum_{n=0}^{\infty} h_n^{(2)}(kr_G) \sum_{m=0}^{n} (C_{nm}(\omega)\cos m\phi_G + D_{nm}(\omega)\sin m\phi_G)\, P_n^m(\cos\theta_G). \tag{9}$$

The new multipole coefficients are then given by

$$\begin{cases} C_{nm}(\omega) = \dfrac{(-1)^m}{2}\sqrt{\dfrac{(n-|m|)!}{(n+|m|)!}}\sqrt{\dfrac{2n+1}{\pi}}\,[G^n_m(\omega)+(-1)^m G^n_{-m}(\omega)] & \text{for } m>0. \\ C_{no}(\omega) = \dfrac{1}{2}\sqrt{\dfrac{2n+1}{\pi}}\,G^n_o(\omega);\ D_{no}(\omega)=0 & \text{for } m=0. \\ D_{nm}(\omega) = \dfrac{i(-1)^m}{2}\sqrt{\dfrac{(n-|m|)!}{(n+|m|)!}}\sqrt{\dfrac{2n+1}{\pi}}\,[G^n_m(\omega)-(-1)^m G^n_{-m}(\omega)] & \text{for } m>0. \end{cases} \tag{10}$$

These coefficients are those of the multipolar expansion in the new coordinate system.

We still have to express the ultraspherical functions in closed form, in order to apply (6). The derivation is made in Appendix B, and the result is

$$P^{mk}_n(\mu) = (-i)^\alpha\, 2^{-n}\sqrt{\frac{s_+!\,s_-!}{t_+!\,t_-!}}\,(1-\mu)^{\alpha/2}(1+\mu)^{\beta/2} \times\sum_{j=0}^{s_-}\binom{s_-+\alpha}{j}\binom{s_-+\beta}{s_--j}(\mu-1)^{s_--j}(\mu+1)^j, \tag{11}$$

where the following definitions hold:

$\alpha = |m-k|$, $\beta = |m+k|$, $s_- = n - ½(\alpha+\beta)$, $s_+ = n + ½(\alpha+\beta)$, $t_- = n - ½(\alpha-\beta)$, $t_+ = n + ½(\alpha-\beta)$.

All these quantities are integers. The formula (11) is then particularly easy to use since it is merely a polynomial. It yields good results, especially for low n (Minster 1973). We are rarely interested in computing more than a few multipoles, and (11) is more than adequate. For larger degrees and orders, Edmonds (1957) gives recursion relations which are easy to use. The reader should be cautioned, however, that Edmonds' choice of Euler angles is slightly different from ours. The adaptation of his results does not pose any major theoretical problem.

Because of the unitarity of the operator T_R, the coefficients appearing in (8) correspond to the inverse rotation of Euler angles $\pi - \Phi_2$, Θ, and $\pi - \Phi_1$. In other words, we can write

$$\overline{T^{km}_n(\Phi_1,\Theta,\Phi_2)} = T^{mk}_n(\pi-\Phi_2,\Theta,\pi-\Phi_1). \tag{12}$$

That this property should be satisfied constitutes a useful check on numerical calculations. Another important check is that, for each n, the power should be conserved under rotation, that is

$$\sum_{m=-n}^{n}|A^n_m|^2 = \sum_{m=-n}^{n}|G^n_m|^2. \tag{13}$$

This result expresses the intuitive fact that the relative excitation of the various multipoles is unchanged by rotation of the coordinate system. This is not the case for a translation as we shall see in the next section.

The derivation of the correspondence between Euler angles Φ_1, Θ, Φ_2, and the usual fault orientation parameters is given in Appendix A.

3 Translation of the coordinate system

Let T be the new coordinate system obtained from S by a translation of vector $\mathbf{d} = (d, \theta_d, \phi_d)$. Again we want to investigate the result of this operation on the multipolar expansion (2). The key to this is the addition theorem for scalar spherical wave functions (Stein 1961; Cruzan 1962; Miller 1964) which can be written (Minster 1973)

$$Z_n(kr_s)\, Y_n^m(\theta_s, \phi_s) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{\ell=|n-\nu|}^{n+\nu} C(\nu, \mu, l|n, m)$$

$$\times\, j_\nu(kr_<)\, Y_\nu^\mu(\theta_<, \phi_<)\, Z_l(kr_>)\, Y_l^{m-\mu}(\theta_>, \phi_>) \tag{14}$$

where Z_n is any spherical Bessel or Hankel function of degree n, and

$$\left.\begin{aligned}(r_<, \theta_<, \phi_<) &= (d, \theta_d, \phi_d)\\ (r_>, \theta_>, \phi_>) &= (r_\mathrm{T}, \theta_\mathrm{T}, \phi_\mathrm{T})\end{aligned}\right\} \quad \text{if } d < r_\mathrm{T}$$

$$\left.\begin{aligned}(r_<, \theta_<, \phi_<) &= (r_\mathrm{T}, \theta_\mathrm{T}, \phi_\mathrm{T})\\ (r_>, \theta_>, \phi_>) &= (d, \theta_d, \phi_d)\end{aligned}\right\} \quad \text{if } d > r_\mathrm{T}.$$

The coefficient appearing in (14) is given by

$$C(\nu, \mu, l|n, m = i^{\nu+l-n} \left(\frac{4\pi(2\nu+1)(2l+1)}{2n+1}\right)^{1/2} (l\nu m - \mu\mu|nm)\,(l\nu 0 0|n 0), \tag{15}$$

and is non-zero only when $l + \nu + n$ is even. The Clebsch–Gordan coefficients appearing in (15) may be evaluated in terms of the Wigner 3-j symbols (Edmonds 1957; Gottfried 1966) by the relation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 - j_2 - m_3}(2j_3+1)^{-1/2}(j_1\, j_2\, m_1\, m_2 | j_3 - m_3). \tag{16}$$

Both the Clebsch–Gordan coefficients and the Wigner 3-j symbols can be evaluated by means of recursions (Edwards 1957). Cruzan (1962) lists a set of particularly useful such relations.

However, in view of the rotation theorem described in the previous section, it is clear that the simpler case when $\mathbf{d}$ is along the polar axis is sufficient for our purposes.

Then θ_d is 0 or π and $\cos\theta_d = \epsilon$. Only the terms for $\mu = 0$ remain, and ϕ_d may be taken to be zero. We have

$$Y_\nu^o(\theta_d, 0) = \sqrt{\frac{2\nu+1}{4\pi}}\, \epsilon^\nu$$

$$\phi_\mathrm{T} = \phi_\mathrm{S}.$$

Therefore (14) becomes, for $r_\mathrm{T} > d$ (which is the usual case in source studies)

$$Z_n(kr_s)\, Y_n^m(\theta_s, \phi_s) = \sum_{\nu=0}^{\infty} \sum_{l=|n-\nu|}^{n+\nu} C_1(\nu, l|n, m)\, j_\nu(kd)\, Z_l(kr_\mathrm{T})\, Y_l^m(\theta_\mathrm{T}, \phi_s) \tag{17}$$

with

$$C_1(\nu, l|n,m) = \epsilon^\nu i^{\nu+l-n}(2\nu+1)\sqrt{\frac{2l+1}{2n+1}}\,(l\nu m0|nm)\,(l\nu 00|n0)$$

$$= \epsilon^\nu(-1)^m i^{\nu+l-n}(2\nu+1)(2n+1)^{1/2}(2l+1)^{1/2}\begin{pmatrix} l & \nu & n \\ m & 0 & -m\end{pmatrix}\begin{pmatrix} l & \nu & n \\ 0 & 0 & 0\end{pmatrix}. \tag{18}$$

The coefficient C_1 vanishes unless $l+n+\nu$ is even, $m \leqslant l$, and $|l-n| \leqslant \nu \leqslant l+n$. Thus the inner sum is a finite one, and it is possible to interchange the order of summation and to reorder the terms so that

$$h_n^{(2)}(kr_s)\,Y_n^m(\theta_s,\phi_s) = \sum_{l=0}^{\infty}\sum_{\nu=|l-n|}^{l+n} C_1(\nu,l|n,m)\,j_\nu(kd)\,h_l^{(2)}(kr_T)\,Y_l^m(\theta_T,\phi_s). \tag{19}$$

We observe that the order m is left unchanged in such a translation. This makes the analysis much more tractable. The series (19) converges uniformly with respect to r_T provided that $r_T > d$ (Friedman & Russek 1954). If we suppose that the series in (2) converges uniformly with respect to r_s in the same region, then by combining these two equations we can write

$$\chi(\mathbf{r}_T,\omega) = \sum_{l=0}^{\infty}\sum_{m=-n}^{n} T_m^l(\omega)\,h_l^{(2)}(kr_T)\,Y_l^m(\theta_T,\phi_T), \tag{20}$$

and the new multipole coefficients are given by

$$T_m^l(\omega) = \sum_{n=0}^{\infty}\sum_{\nu=|l-n|}^{l+n} A_m^n(\omega)\,C_1(\nu,l|n,m)\,j_\nu(kd). \tag{21}$$

To complete the transformation, we rewrite (20) in a form similar to (9). The new multipole coefficients are then given by equations identical to (10) where T_m^l replaces G_m^n.

In most cases, when the source is of the double couple or of the quadrupole type (e.g. Randall 1966; Archambeau 1968), only one value of n ($n = 2$) is present in the initial expansion (2). In that case, there is no convergence problem since we have only a finite sum in (21). For the case where $r_T < d$, the same analysis can be easily duplicated by interchanging the roles of $\mathbf{r}_T$ and $\mathbf{d}$ from the beginning.

It is clear that a translation does not preserve the power contained in a multipole of a given degree. This is intuitively understandable since a translation does not preserve spherical symmetry. In fact, if a pure double-couple source is expanded in the coordinate system satisfying its symmetry, the expansion will be a pure quadrupole. But if it is a shallow source and we want to expand it in a geocentric system, it is obvious that a large number of very high order multipoles will be necessary to represent it: when seen from the centre of the Earth this source is very localized, and seems like a singularity at the surface.

Note that if $d = 0$, there is no translation, then $\nu = 0$ is the only term present and we have $l = n$, thus

$$h_n^{(2)}(kr_s)\,Y_n^m(\theta_s,\phi_s) = C_1(0,n|n,m)\,h_n^{(2)}(kr_s)\,Y_n^m(\theta_s,\phi_s). \tag{22}$$

But

$$C_1(0,n|n,m) = (n0m0|nm)\,(n000|n0) = 1$$

so that (22) is a proper equality.

Conclusion

We have derived expressions for the transformation of a multipolar expansion under an arbitrary proper three-dimensional coordinate transformation. These expressions prove particularly handy for solutions of wave propagation problems when the source has arbitrary orientation. Expressions for the Euler angles in terms of the usual geological parameters 'strike', 'dip', and 'plunge' are given in Appendix A. Expressions for the most commonly called for 3-*j* symbols to use in conjunction with (16) are listed in Appendix C. Some geophysical applications of these results are contained in representations of the radiation fields due to the relaxation of stress around propagating failure zones in prestressed media (e.g. Minster 1973; Minster & Archambeau, in preparation). Since the results presented here describe the behaviour of eigenfunctions of the wave equation under transformation of the coordinates, one sees readily that applications are not limited to problems in the theory of elasticity, but can also be found in a wide range of geophysical theories as well.

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Appendix A

Derivation of Euler angles from fault orientation parameters

As described in the text, the most convenient coordinate system to represent the radiation field from a propagating rupture is that one with the z-axis along the direction of propagation. We have shown how to transform the radiation field under a rotation of the reference frame, if the Euler angles are known. We now derive these Euler angles to transform the coordinate system to the local geographical system, described on Fig. 2. The z-axis is along the local vertical, the x-axis in a northerly direction.

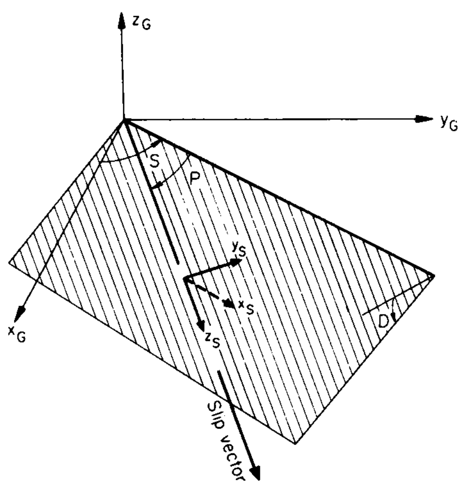

Figure 2. Fault geometry relating the source coordinate system to the geographical coordinate system. S, D, P, are the strike, dip, and plunge angles respectively.

The fault geometry can be described by its strike, dip, and plunge, denoted S, D, and P respectively. We choose the convention that the strike be measured counterclockwise from the north so that $-\pi \leqslant S \leqslant \pi$, or $0 \leqslant S \leqslant 2\pi$. The dip can then be measured clockwise from the horizontal by a vertical observer at the hypocentre so that $0 \leqslant D \leqslant \pi$. The same observer measures the plunge downward from the horizontal, thus $-\pi/2 \leqslant P \leqslant \pi/2$. The unit vectors

$\mathbf{x}_G$, $\mathbf{y}_G$, $\mathbf{z}_G$ are then transformed into the unit vectors $\mathbf{s}_S$, $\mathbf{y}_S$, $\mathbf{z}_S$, by the (orthogonal) rotation matrix

$$\mathbf{T} = \begin{bmatrix} -\sin S \sin D & -\sin S \cos D \cos P + \cos S \sin P & \sin S \cos D \sin P + \cos S \cos P \\ \sin D \cos S & \cos S \cos D \cos P + \sin S \sin P & -\cos S \cos D \sin P + \sin S \cos P \\ -\cos D & \sin D \cos P & -\sin D \sin P \end{bmatrix}. \tag{A1}$$

The columns of $\mathbf{T}$ are the components of $\mathbf{x}_S$, $\mathbf{y}_S$, $\mathbf{z}_S$ in the $\mathbf{x}_G$, $\mathbf{y}_G$, $\mathbf{z}_G$, system. $\mathbf{T}$ can then be written equivalently in terms of the Euler angles Φ_1, Θ, Φ_2 described on Fig. 1 (Gel'fand 1963). Calling the new matrix $\mathbf{R}$, we have

$$\mathbf{R} = \begin{bmatrix} \cos\Phi_1 \cos\Phi_2 - \cos\Theta \sin\Phi_1 \sin\Phi_2 & -\cos\Phi_1 \sin\Phi_2 - \cos\Theta \sin\Phi_1 \cos\Phi_2 & \sin\Theta \sin\Phi_1 \\ \sin\Phi_1 \cos\Phi_2 + \cos\Theta \cos\Phi_1 \sin\Phi_2 & -\sin\Phi_1 \sin\Phi_2 + \cos\Theta \cos\Phi_1 \cos\Phi_2 & -\sin\Theta \cos\Phi_1 \\ \sin\Phi_2 \sin\Theta & \sin\Theta \cos\Phi_2 & \cos\Theta \end{bmatrix}. \tag{A2}$$

Here $0 \leqslant \Phi_1 \leqslant 2\pi$, $0 \leqslant \Theta \leqslant \pi$, $0 \leqslant \Phi_2 \leqslant 2\pi$.

We obtain the relation between the fault orientation parameters and the Euler angles by simple identification of $\mathbf{T}$ and $\mathbf{R}$.

(1) $T_{33} = \pm 1$.

In that case the transformation is a simple rotation of angle Φ_1 around the z-axis, thus

$$\left.\begin{aligned} \Phi_1 &= \cos^{-1}(T_{11}); \quad \Theta = \begin{cases} 0 \\ \pi \end{cases}; \quad \Phi_2 = 0 \qquad \text{if } T_{21} > 0 \\ \Phi_1 &= 2\pi - \cos^{-1}(T_{11}); \quad \Theta = \begin{cases} 0 \\ \pi \end{cases}; \quad \Phi_2 = 0 \qquad \text{if } T_{21} < 0 \end{aligned}\right\}. \tag{A3}$$

(2) $T_{33} \neq \pm 1$.

Then because $0 < \Theta < \pi$

$$\Theta = \cos^{-1}(T_{33}) \tag{A4}$$

$\sin\Theta = (1 - T_{33}^2)^{1/2}$ is a positive quantity. Furthermore we have

$$\sin\Phi_1 = \frac{T_{13}}{\sin\Theta}, \quad \cos\Phi_1 = -\frac{T_{23}}{\sin\Theta},$$

$$\sin\Phi_2 = \frac{T_{31}}{\sin\Theta}, \quad \cos\Phi_2 = \frac{T_{32}}{\sin\Theta}.$$

Therefore

$$\left.\begin{aligned} \Phi_1 &= \cos^{-1}\frac{-T_{23}}{(1 - T_{33}^2)^{1/2}} \qquad \text{if } T_{13} > 0 \\ \Phi_1 &= 2\pi - \cos^{-1}\frac{T_{32}}{(1 - T_{33}^2)^{1/2}} \qquad \text{if } T_{13} < 0 \end{aligned}\right\} \tag{A5}$$

and

$$\left.\begin{aligned}\Phi_2 &= \cos^{-1}\frac{T_{32}}{(1-T_{33}^2)^{1/2}} && \text{if } T_{31}>0\\ \Phi_2 &= 2\pi-\cos^{-1}\frac{T_{32}}{(1-T_{33}^2)^{1/2}} && \text{if } T_{31}<0\end{aligned}\right\}. \tag{A6}$$

The Euler angles thus obtained transform the geographical coordinates into the source coordinates. The inverse rotation may be obtained by replacing the matrix T_{ij} by its transposed T_{ji} in the results above; the Euler angles for the inverse rotation are $\pi - \Phi_2, \Theta, \pi - \Phi_1$.

Appendix B

Ultraspherical functions and Jacobi polynomials

The formulae describing the transformation of multipolar expansions under rotations of the coordinate system are given in the text. They involve ultraspherical functions for which we now derive a simple closed form for the cases of interest to us.

Gel'fand (1963) gives the following analytical expressions for the ultraspherical functions.

$$\mathrm{P}_n^{m\kappa}(\mu) = F(1-\mu)^{-(\kappa-m)/2}(1+\mu)^{-(\kappa+m)/2}\frac{d^{n-\kappa}}{d\mu^{n-\kappa}}[(1-\mu)^{n-m}(1+\mu)^{n+m}] \tag{B1}$$

where

$$F = \frac{(-1)^{n-m}i^{\kappa-m}}{2^n(n-m)!}\sqrt{\frac{(n-m)!\,(n+\kappa)!}{(n+m)!\,(n-\kappa)!}}.$$

These functions possess the following symmetry properties (Gel'fand 1963)

$$\left\{\begin{aligned}&\mathrm{P}_n^{m\kappa}(\mu)=\mathrm{P}_n^{\kappa m}(\mu)\\ &\mathrm{P}_n^{-m-\kappa}(\mu)=\mathrm{P}_n^{m\kappa}(\mu)\end{aligned}\right. \tag{B2}$$

and therefore depend only on the values of $|m+\kappa|$ and $|m-\kappa|$. One is thus led to define

$\alpha = |m-\kappa|, \beta = |m+\kappa|, s_- = n - \tfrac{1}{2}(\alpha+\beta), s_+ = n + \tfrac{1}{2}(\alpha+\beta), t_- = n - \tfrac{1}{2}(\alpha-\beta),$
$t_+ = n + \tfrac{1}{2}(\alpha-\beta),$

where all of these quantities are integers.

By identification we can rewrite (B1) as

$$\mathrm{P}_n^{m\kappa}(\mu) = K(\alpha, n, s_+, s_-, t_+, t_-)\,.\,(1-\mu)^{\alpha/2}(1+\mu)^{\beta/2}P_{s_-}^{\alpha\beta}(\mu) \tag{B3}$$

where K is a constant and $P_{s_-}^{\alpha\beta}(\mu)$ are the Jacobi polynomials in Erdelyi's notation

$$P_{s_-}^{\alpha\beta}(\mu) = \frac{(-1)^{s_-}}{2^{s_-}.\,s_-!}(1-\mu)^{-\alpha}(1+\mu)^{-\beta}\frac{d^{s_-}}{d\mu^{s_-}}[(1-\mu)^{s_-+\alpha}(1+\mu)^{s_-+\beta}]. \tag{B4}$$

Since the indices are integers, (B4) can be rewritten in closed form

$$P_{s_-}^{\alpha\beta}(\mu) = 2^{-s_-}\sum_{j=0}^{s_-}\binom{s_-+\alpha}{j}\binom{s_-+\beta}{s_--j}(\mu-1)^{s_--j}(\mu+1)^j. \tag{B5}$$

We evaluate K by simple identification

$$\frac{1}{K} = i^{\alpha} 2^{n-s_-} \sqrt{\frac{t_+!\,t_-!}{s_+!\,s_-!}}. \tag{B6}$$

Equations (B3), (B5) and (B6) are then combined to yield

$$\mathrm{P}_n^{m\kappa}(\mu) = (-i)^{\alpha} 2^{-n} \sqrt{\frac{s_+!\,s_-!}{t_+!\,t_-!}} \sum_{j=0}^{s_-} \binom{s_- + \alpha}{j} \binom{s_- + \beta}{s_- - j} (\mu - 1)^{s_- - j} (\mu + 1)^j$$
$$\times (1-\mu)^{\alpha/2} (1+\mu)^{\beta/2}. \tag{B7}$$

Equation (B7) provides a closed form for the ultraspherical function which is particularly suitable for computation on digital machines.

Formula (B7) is a polynomial involving few terms, especially for low degree n. It is thus particularly convenient for numerical computations in those cases. For seismological problems one is rarely interested in considering more than a few multipoles and (B7) is adequate. For larger degrees and orders useful recurrence relations are given by Edmonds (1957).

Appendix C

Commonly used 3-j symbols

Vector coupling coefficients, or 3-j symbols are used in the addition theorem for spherical wave functions. The table given below provides closed forms for the most commonly used 3-j symbols in source theory. This table is derived from the closed forms tabulated by Edmonds (1957).

(a) $\nu = l - 2$

$$\begin{pmatrix} l & l-2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \nu+2 & \nu & 2 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\nu} (\nu+2)(\nu+1) \sqrt{\frac{6(2\nu)!}{(2\nu+5)!}}$$

$$\begin{pmatrix} l & l-2 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \nu+2 & \nu & 2 \\ 1 & 0 & -1 \end{pmatrix} = 2(-1)^{\nu+1} (\nu+1) \sqrt{(\nu+2)(\nu+3)} \sqrt{\frac{(2\nu)!}{(2\nu+5)!}}$$

$$\begin{pmatrix} l & l-2 & 2 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} \nu+2 & \nu & 2 \\ 2 & 0 & -2 \end{pmatrix} = (-1)^{\nu} \sqrt{(\nu+1)(\nu+2)(\nu+3)(\nu+4)} \sqrt{\frac{(2\nu)!}{(2\nu+5)!}}.$$

(b) $\nu = l$

$$\begin{pmatrix} l & l & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \nu & \nu & 2 \\ 0 & 0 & 0 \end{pmatrix} = 2(-1)^{\nu+1} \nu(\nu+1) \sqrt{\frac{(2\nu-2)!}{(2\nu+3)!}}$$

$$\begin{pmatrix} l & l & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \nu & \nu & 2 \\ 1 & 0 & -1 \end{pmatrix} = (-1)^{\nu} \sqrt{\nu(\nu+1)} \sqrt{\frac{6(2\nu-2)!}{(2\nu+3)!}}$$

$$\begin{pmatrix} l & l & 2 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} \nu & \nu & 2 \\ 2 & 0 & -2 \end{pmatrix} = (-1)^{\nu} \sqrt{(\nu-1)\nu(\nu+1)(\nu+2)} \sqrt{\frac{6(2\nu-2)!}{(2\nu+3)!}}.$$

(c) $\nu = l + 2$

$$\begin{pmatrix} l & l+2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \nu-2 & \nu & 2 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^\nu \nu(\nu-1) \sqrt{\frac{6(2\nu-4)!}{(2\nu+1)!}}$$

$$\begin{pmatrix} l & l+2 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \nu-2 & \nu & 2 \\ 1 & 0 & -1 \end{pmatrix} = 2(-1)^\nu \nu\sqrt{(\nu-1)(\nu-2)} \sqrt{\frac{(2\nu-4)!}{(2\nu+1)!}}$$

$$\begin{pmatrix} l & l+2 & 2 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} \nu-2 & \nu & 2 \\ 2 & 0 & -2 \end{pmatrix} = (-1)^\nu \sqrt{(\nu-3)(\nu-2)(\nu-1)\nu} \sqrt{\frac{(2\nu-4)!}{(2\nu+1)!}}.$$